

Several results from numerical investigation of nonlinear waves connected to blood flow in an elastic tube of variable radius

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Abstract

We investigate flow of incompressible fluid in a cylindrical tube with elastic walls. The radius of the tube may change along its length. The discussed problem is connected to the blood flow in large human arteries and especially to nonlinear wave propagation due to the pulsations of the heart. The long-wave approximation for modeling of waves in blood is applied. The obtained model Korteweg-deVries equation possessing a variable coefficient is reduced to a nonlinear dynamical system of 3 first order differential equations. The low probability of arising of a solitary wave is shown. Periodic wave solutions of the model system of equations are studied and it is shown that the waves that are consequence of the irregular heart pulsations may be modeled by a sequence of parts of such periodic wave solutions.

1 Introduction

Nonlinear phenomena are usual for physics and especially for fluid mechanics [1] - [10]. One of the most interesting nonlinear phenomena are the nonlinear waves propagating in various media [11] - [17] and especially in fluids [18] - [21]. In this paper we shall discuss nonlinear waves connected to blood flow

in large arteries [22] - [24]. There exist many kinds of possible methodologies to investigate traveling waves of the model nonlinear partial differential equations. One of them is to reduce the corresponding nonlinear PDE to a system of nonlinear ordinary differential equations and then to investigate the obtained system numerically. Another is to apply methods for obtaining (exact) solutions of the studied nonlinear partial differential equation [25] - [29]. We shall use the first of the above methodologies in this paper.

Fluid flow connected to spreading of pressure pulses in elastic tubes is of interest for arterial mechanics of large arteries [30] - [36]. In this case the nonlinearities are important for the study of the flow and because of this one has to use non-linear model differential equations. In the mathematical models arteries usually are treated as circularly cylindrical long homogeneous isotropic tubes which length is much larger than its radius (i.e. the corresponding tube can be considered as thin tube with respect to the ratio length/thickness). Below we shall study the propagation of non-linear waves in a fluid-filled long elastic tube with variable radius. The fluid will be incompressible (this a reasonable assumption for the case of blood flow in large arteries) and the tube will be isotropic, inhomogeneous and prestressed. The model equations of the flow will be reduced to a variant of the Korteweg-deVries equation with one variable coefficient. This nonlinear partial differential equation will be further reduced to a system of 3 nonlinear ordinary differential equations for the case of traveling waves. The system of nonlinear ordinary differential equations will be studied numerically.

The organization of the paper is as follows. In Sect. 2. we discuss the model equations for the blood flow in a large artery. In Sect.3 by application of the long-wave approximation the model equation will be reduced to a forced Korteweg-deVries equation and this equation will be further reduced to a system of 3 nonlinear ordinary differential equation. In Sect. 4 we perform a numerical study of the system of nonlinear ODEs. Several concluding remarks are summarized in Sect. 5.

2 Mathematical formulation of the model

For our study we shall use the nonlinear model presented in [30]. The model has two parts: equations of the elastic tube and equation of the fluid in the tube. First we describe the model equations of the elastic tube. We set a cylindrical polar co-ordinate system with axial axis coinciding to the axis of

the (straight) studied tube. We denote the base vectors of this co-ordinate system as $\vec{e}_r, \vec{e}_\theta, \vec{e}_z$. R_0 is the radius of the tube before the stretching. The tube is assumed to be tapered (with small tapering angle Φ). Z^* is the axial co-ordinate along the axis of the elastic tube. Because of the tapering the radial coordinate of the tube point with axial co-ordinate Z will be $R_0 + Z^* \sin \Phi \approx R_0 + Z^* \Phi$. Thus the initial coordinate of the point will be $\vec{R} = (R_0 + Z^* \Phi) \vec{e}_r + Z^* \vec{e}_z$. In this case of initial conditions (with no stretching) the length of the elements of the meridional and circumferential curves at the point with coordinate \vec{R} are as follows. The meridional curve is a straight line. If the change of the Z^* co-ordinate is dZ^* then the change of the radius is $dR = dZ^* \sin \Phi \approx dZ^* \Phi$. Then the length of the element of the meridional curve will be $dS_Z^2 = dZ^{*2} + dZ^{*2} \Phi^2$ and then

$$dS_Z = (1 + \Phi^2)^{1/2} dZ^* \quad (1)$$

The length of the circumferential curve is given by the relationship $dS_\theta/R = \sin(d\theta) \approx d\theta$ where $R = R_0 + Z^* \Phi$. Then the length of the circumferential curve is

$$dS_\theta = (R_0 + Z^* \Phi) d\theta \quad (2)$$

We assume that there is axial stretch of the tube. After the stretching the axial co-ordinate becomes $z^* = \lambda_z Z^*$ where λ_z is the axial stretch ratio (for the case without stretching $\lambda_z = 1$). In addition there is axially-dependent static pressure $P_0(Z^*)$ imposed on the tube (it has to be understood as the end of the diastolic pressure). Let r_0 be the deformed radius of the tube at the origin of coordinate system Then the coordinate of a point of the tube is

$$\vec{r}_0(z^*) = [r_0 + \phi z^*] \vec{e}_r + z^* \vec{e}_z \quad (3)$$

where ϕ is the tapering angle after imposing the pressure (the tube responds to this pressure by changing its form and Eq. (3) is for the case when the form of the tube after the imposing of the pressure remains cone). If the form of the tube do not remains cone then we have to introduce the function $f^*(z)$ which characterizes the radius change and instead of Eq.(3) we shall have

$$\vec{r}_0(z^*) = [r_0 + f(z^*)] \vec{e}_r + z^* \vec{e}_z \quad (4)$$

Now the lengths of the elementary meridional and circumferential curve elements (for the case when the deformed form of the tube doesn't remain cone)

are as follows. The length of the meridional curve element is:

$$ds_z^0 = \left[1 + \left(\frac{\partial f^*}{\partial z^*} \right)^2 \right]^{1/2} dz^* \quad (5)$$

The length of the circumferential curve element is:

$$ds_\theta^0 = (r_0 + f^*) d\theta \quad (6)$$

Additional (dynamical) deformation of the tube arises from the presence of fluid (blood) flow. This deformation depends on the coordinate z^* and on the time t^* . Let us denote this deformation as $u^*(z^*, t^*)$. Then the coordinate of the point of the tube becomes

$$\vec{r}_0(z^*) = [r_0 + f(z^*) + u^*(z^*, t^*)] \vec{e}_r + z^* \vec{e}_z \quad (7)$$

In this case the lengths of the elementary meridional and circumferential tubes are as follows. The length of the meridional curve element is:

$$ds_z^0 = \left[1 + \left(\frac{\partial f^*}{\partial z^*} + \frac{\partial u^*}{\partial z^*} \right)^2 \right]^{1/2} dz^* \quad (8)$$

The length of the circumferential curve element is:

$$ds_\theta^0 = (r_0 + f^* + u^*) d\theta \quad (9)$$

The corresponding stretch ratios are

$$\lambda_1 = \frac{ds_z}{dS_z}; \quad \lambda_2 = \frac{ds_\theta}{dS_\theta} \quad (10)$$

After the static and dynamic deformation (when the tube does not remains cone after the static deformation)

$$\lambda_1 = \lambda_z \frac{[(1 + [(\partial f^*/\partial z^*) + (\partial u^*/\partial z^*)]^2)^{1/2}]}{(1 + \Phi^2)^{1/2}}; \quad \lambda_2 = \lambda_z \frac{r_0 + f^* + u^*}{\lambda_z R_0 + z^* \Phi} \quad (11)$$

For the case when $\Phi = 0$ (i.e. the tube before applying the static pressure is cylinder and not a cone):

$$\lambda_1 = \lambda_z [(1 + [(\partial f^*/\partial z^*) + (\partial u^*/\partial z^*)]^2)^{1/2}]; \quad \lambda_2 = \lambda_z \frac{r_0 + f^* + u^*}{\lambda_z R_0} \quad (12)$$

The relationship for λ_2 can be written as follows

$$\lambda_2 = \frac{r_0}{R_0} + \frac{f^* + u^*}{R_0} = \lambda_\theta + \frac{f^* + u^*}{R_0} \quad (13)$$

where $\lambda_\theta = r_0/R_0$.

In this case the tube is deformed and in the general case the unit normal vector \vec{n} does not coincide to the unit vector \vec{e}_r . The unit tangential vector \vec{t} of the curved surface also do not coincide to the vector \vec{e}_z . The unit normal and tangential vectors can be expressed by the unit vectors \vec{e}_r, \vec{e}_z . When we take into account that

$$[1 + [(\partial f^*/\partial z^*) + (\partial u^*/\partial z^*)]^2]^{1/2} = \frac{\lambda_1}{\lambda_z} \quad (14)$$

then

$$\vec{t} = \frac{[(\partial f^*/\partial z^*) + (\partial u^*/\partial z^*)]\vec{e}_r + \vec{e}_z}{(1 + [(\partial f^*/\partial z^*) + (\partial u^*/\partial z^*)]^2)^{1/2}} = \frac{\lambda_z}{\lambda_1} [(\partial f^*/\partial z^*) + (\partial u^*/\partial z^*)]\vec{e}_r + \vec{e}_z \quad (15)$$

$$\vec{n} = \frac{\vec{e}_r - [(\partial f^*/\partial z^*) + (\partial u^*/\partial z^*)]\vec{e}_z}{(1 + [(\partial f^*/\partial z^*) + (\partial u^*/\partial z^*)]^2)^{1/2}} = \frac{\lambda_z}{\lambda_1} [\vec{e}_r - [(\partial f^*/\partial z^*) + (\partial u^*/\partial z^*)]\vec{e}_z] \quad (16)$$

4 forces are responsible for the movement of a fluid element. The first force is the force of movement in radial direction due to the existing pressure difference. The second force is the force along the meridional curve. The third force is the force acting along the circumferential curve. The force connected to the movement of the tube element is equal to the mass of the tube element multiplied by its acceleration. Let the tube thickness before the static deformation be H . The thickness of the tube after the static deformation will be h . Then the mass of the tube element is approximately $\rho_w h R d\theta dz$ where ρ_w is the mass density of the material of the tube. The acceleration of the element is equal to $\frac{\partial^2 u^*}{\partial t^{*2}}$. Thus this term of the equation of balance of forces becomes $\rho_w h R d\theta dz \frac{\partial^2 u^*}{\partial t^{*2}}$. The tube thickness h after the static deformation can be expressed by the tube thickness H before the static deformation. The assumption is that the material is incompressible. This leads to

$$\lambda_1 \lambda_2 h = H \rightarrow h = \frac{H}{\lambda_1 \lambda_2} \quad (17)$$

In our case (when axial stretching exists) the initial radius R_0 for tube with length dl transforms to radius R for a tube of length $\lambda_z dl$. Assuming

that the area remains the same we obtain $R = R_0/\lambda_z$. Thus for the force per unit $d\theta dz$ we obtain $\rho_w H(R_0/\lambda_z) \frac{\partial^2 u^*}{\partial t^{*2}}$.

The second force is the pressure force that acts on the tube element. This force is equal to the pressure difference $P - P_e$ (where P is the pressure in the tube and P_e is the external pressure) multiplied by the surface of the tube element that is $Rd\theta dz$. Then this force becomes $(P - P_e) \cos \phi (Rd\theta dl)$. In our case $P_e = P_0$ and $P - P_e = P^*$. The co-ordinate R connected to the element of the tube is $R = r_0 + f^* + u^*$. And the entire force per unit $d\theta dz$ becomes $P^*(r_0 + f^* + u^*)$.

The remaining two forces are connected to the membrane forces T_1 and T_2 that act along the circumferential and meridional curves of the tube. For unit $d\theta dz$ the force acting along the meridional curve is $F_2 \vec{t}$. Its vertical component is $T_2 = F_2 \vec{t} \cdot \vec{n}$ from where $T_2 = -F_2 \lambda_z / \lambda_1$ and $F_2 = -T_2 \lambda_1 / \lambda_z$. The force F_1 is [30, 37]

$$F_1 = \frac{\partial}{\partial z^*} \left[\frac{\lambda_z}{\lambda_1} (r_0 + f^* + u^*) \left(\frac{\partial f^*}{\partial z} + \frac{\partial u^*}{\partial z} \right) T_1 \right] \quad (18)$$

The balance of the above 4 forces is

$$-\frac{\lambda_1}{\lambda_z} T_2 + \frac{\partial}{\partial z^*} \left\{ \frac{\lambda_z (r_0 + f^* + u^*) \left(\frac{\partial f^*}{\partial z^*} + \frac{\partial u^*}{\partial z^*} \right) T_1}{\lambda_1} \right\} + P^*(r_0 + f^* + u^*) = \rho_w H \frac{R_0}{\lambda_z} \frac{\partial^2 u^*}{\partial t^{*2}} \quad (19)$$

Let $\mu(z^*)$ be the variable shear modulus of the tube material and $\mu(z^*)\Pi$ be the strain energy function of the membrane. Then the membrane forces $T_{1,2}$ can be written as

$$T_1 = \frac{H}{\lambda_2} \mu(z^*) \frac{\partial \Pi}{\partial \lambda_1}; \quad T_2 = \frac{H}{\lambda_1} \mu(z^*) \frac{\partial \Pi}{\partial \lambda_2} \quad (20)$$

After a substitution of Eq.(20) in Eq. (19) we obtain the pressure P^* as a function of u^* and its derivatives.

$$P^* = \frac{\rho_w H R_0}{\lambda_z (r_0 + f^* + u^*)} \frac{\partial^2 u^*}{\partial t^{*2}} + \frac{\mu(z^*)}{\lambda_z (r_0 + f^* + u^*)} \frac{\partial \Pi}{\partial \lambda_2} - \frac{\lambda_z R_0}{(r_0 + f^* + u^*)} \frac{\partial}{\partial z^*} \left[\frac{\mu(z^*)}{\lambda_1} \left(\frac{\partial f^*}{\partial z^*} + \frac{\partial y^*}{\partial z^*} \right) \frac{\partial \Pi}{\partial \lambda_1} \right] \quad (21)$$

The model equations of the fluid in the tube are as follows. The blood in a large arteries can be approximated by a Newtonian fluid with respect to its flow (this is not the case for blood flow in small arteries). In addition the viscosity of the blood may be neglected as a first approximation [38, 39] and the variation of the quantities with the radial coordinate will be disregarded too. What remains from the averaged Navier-Stokes equations in cylindrical coordinates is

$$\frac{\partial w^*}{\partial t^*} + w^* \frac{\partial w^*}{\partial z^*} + \frac{1}{\rho_f} \frac{\partial P^*}{\partial z^*} = 0 \quad (22)$$

where ρ_f is the density of the fluid and W^* is the axial fluid velocity. In addition

$$\frac{\partial A^*}{\partial t^*} + \frac{\partial}{\partial z^*}(w^* A^*) = 0 \quad (23)$$

where A^* is the cross-sectional area of the tube. This area is

$$A^* = \pi(r_0 + f^* + u^*)^2 \quad (24)$$

3 Non-dimensionalization of the equations and long-wave approximation

The following nondimensional quantities t , z , R_0 , u , w , m , p , c_0 and $E(z)$ are introduced as follows

$$\begin{aligned} t^* &= t \left(\frac{R_0}{c_0} \right); \quad c_0^2 = \frac{\mu_0 H}{\rho_f R_0}; \quad z^* = R_0 z; \quad r_0 = \lambda_\theta R_0 \\ u^* &= R_0 u, \quad m = \frac{\rho_w H}{\rho_f R_0}; \quad w^* = w c_0; \quad P^* = p \rho_f c_0; \quad \mu = \mu_0 E(z) \end{aligned} \quad (25)$$

The model equations for the unknown functions u , w and p in dimensionless coordinates are:

$$\frac{\partial w}{\partial t} + w \frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = 0 \quad (26)$$

$$2 \frac{\partial u}{\partial z} + 2w \left(\frac{\partial f}{\partial z} + \frac{\partial u}{\partial z} \right) + (\lambda_\theta + f + u) \frac{\partial w}{\partial z} = 0 \quad (27)$$

$$p = \frac{m}{\lambda_z(\lambda_\theta + f + u)} \frac{\partial^2 y}{\partial t^2} + \frac{E(z)}{\lambda_z(\lambda_\theta + f + u)} \frac{\partial \Pi}{\partial \lambda_2} - \frac{\lambda_z}{(\lambda_\theta + f + u)} \left[\frac{E(z)(\frac{\partial f}{\partial z} + \frac{\partial u}{\partial z})}{\lambda_1} \frac{\partial \Pi}{\partial \lambda_1} \right] \quad (28)$$

In order to proceed further we shall consider the case of propagation of small (but finite) amplitude waves in an inhomogeneous thin elastic tube of variable radius and filled with Newtonian fluid. We assume that ϵ is a small parameter and introduce the following coordinates

$$\xi = \epsilon^{1/2}(z - gt); \quad \tau = \epsilon^{3/2}z \quad (29)$$

From here $z = \tau\epsilon^{-2/3}$ and we can use the notations $h(\epsilon, \tau) = f(z)$ and $\hat{E}(\tau, \epsilon) = E(z)$.

The next step is to expand u , w , p , h and \hat{E} in series of the small parameter ϵ

$$\begin{aligned} u &= \epsilon u_1(\xi, \tau) + \epsilon^2 u_2(\xi, \tau) + \dots \\ w &= \epsilon w_1(\xi, \tau) + \epsilon^2 w_2(\xi, \tau) + \dots \\ p &= p_0 + \epsilon p_1(\xi, \tau) + \epsilon^2 p_2(\xi, \tau) + \dots \\ h(\epsilon, \tau) &= 1 + \epsilon h_1(\tau) + \dots \\ \hat{E}(\epsilon, \tau) &= 1 + \epsilon E_1(\tau) + \dots \end{aligned} \quad (30)$$

Let $U(\xi, \tau) = u_1$. From the systems of equations of orders ϵ and ϵ^2 we obtain for U the partial differential equation

$$\frac{\partial U}{\partial \tau} + \mu_2 U \frac{\partial U}{\partial \xi} + \mu_2(\tau) \frac{\partial U}{\partial \xi} + \mu_3 \frac{\partial^3 U}{\partial \xi^3} = 0 \quad (31)$$

The other unknown functions are

$$\begin{aligned} w_1 &= 2 \frac{g}{\lambda_0} [U + \hat{w}_1(\tau)] \\ \hat{w}_1(\tau) &= - \left(h_1 + \frac{\beta_0}{\beta_1} E_1 \right) \\ \beta_0 &= \frac{1}{\lambda_\theta \lambda_z} \frac{\partial \Pi}{\partial \lambda_\theta} \Big|_{u=0}; \quad \beta_1 = \frac{1}{\lambda_\theta \lambda_z} \frac{\partial^2 \Pi}{\partial \lambda_\theta^2} \Big|_{u=0} - \frac{\beta_0}{\lambda_\theta} \\ g^2 &= \frac{\beta_1}{2\lambda_\theta} \\ p_1 &= 2 \frac{g^2}{\lambda_\theta} (h_1 + U) + \beta_0 E_1 \end{aligned} \quad (32)$$

and $\mu_{1,2,3}$ are as follows

$$\begin{aligned}
\mu_1 &= \frac{5}{2\lambda_\theta + \frac{\beta_2}{\beta_1}} \\
\beta_2 &= \frac{1}{2\lambda_\theta\lambda_z} \frac{\partial^3 \Pi}{\partial \lambda_\theta^3} \Big|_{u=0} + \frac{\beta_0}{\lambda_\theta^2} - \frac{\beta_1}{\lambda_\theta} \\
\mu_2(\tau) &= \left(\frac{\beta_2}{\beta_1} - \frac{3}{2\lambda_\theta} \right) h_1(\theta) + \left(\frac{1}{2} - \frac{2\beta_0}{\beta_1\lambda_\theta} E_1(\tau) \right) \\
\mu_3 &= \frac{1}{\lambda_z\lambda_\theta} \left(\frac{m}{4\lambda_\theta} - \frac{\alpha_0}{2\beta_1} \right)
\end{aligned} \tag{33}$$

Finally we have to deal with the variable coefficient $\mu_2(\theta)$ in Eq.(31). We introduce the new coordinate

$$\eta = \xi + \tau - \int_0^\tau ds \mu_2(s) \tag{34}$$

The substitution of Eq.(34) in Eq.(31) lead to the equation

$$\frac{dU}{d\eta} + \mu_1 U \frac{dU}{d\eta} + \mu_3 \frac{d^3 U}{d\eta^3} = 0 \tag{35}$$

Let $V = dU/d\eta$ and $W = dV/d\eta$. Then Eq.(35) is reduced to the following system of 3 equations for the unknown functions U, V, E :

$$\begin{aligned}
\frac{dU}{d\eta} &= V, \\
\frac{dV}{d\eta} &= W, \\
\mu_3 \frac{dW}{d\eta} &= -V(1 + \mu_1 U)
\end{aligned} \tag{36}$$

We remember that U is connected to the deformation of the tube due to the presence of fluid and this quantity is the main quantity of interest for us in this paper.

4 Numerical results

Eq.(31) possesses a solitary wave solution. This solitary wave solution is connected to the solitary wave solution of the classical Korteweg-deVries

equation

$$\frac{\partial A}{\partial t} + \alpha A \frac{\partial A}{\partial x} + \beta \frac{\partial^3 A}{\partial x^3} = 0 \quad (37)$$

Now let

$$A^* = \alpha A/6; \quad x^* = x/\beta^{1/2}; \quad t^* = t/\beta^{1/2} \quad (38)$$

The result of substitution of Eq.(38) in Eq.(37) (we drop the *-s) is

$$\frac{\partial A}{\partial t} + 6A \frac{\partial A}{\partial x} + \frac{\partial^3 A}{\partial x^3} = 0 \quad (39)$$

Let us search for travelling-wave solutions of Eq.(39) of the kind $A(x, t) = A(\zeta) = A(x - vt)$. Eq.(39) becomes

$$\frac{dA}{d\zeta} - \frac{6}{v} A \frac{dA}{d\zeta} - \frac{1}{v} \frac{d^3 A}{d\zeta^3} = 0 \quad (40)$$

which is the same as Eq.(35) when $\mu_1 = -6/v$ and $\mu_3 = -1/v$. The solitary wave solution of Eq.(32) is

$$U(\eta) = \frac{v}{2} \text{sech}^2 \left(\frac{v^{1/2}}{2} \eta \right) \quad (41)$$

The realization of this solution for the case of blood flow in large arteries however has low probability because of two reasons. First the existence of the solution (41) requires a relationship between μ_1 and μ_3 (namely $\mu_1 = 6\mu_3$) that may not be present in the practical situations. And second the realization of the solution (41) requires specific boundary conditions. The derivatives of (41) are as follows

$$\begin{aligned} \frac{dU}{d\eta} &= -\frac{v^{3/2} \sinh(v^{1/2}\eta/2)}{\cosh^3(v^{1/2}\eta/2)}; \quad \frac{d^2U}{d\eta^2} = \frac{v^2[2 \cosh^2(v^{1/2}\eta/2) - 3]}{4 \cosh^4(v^{1/2}\eta/2)} \\ \frac{d^3U}{d\eta^3} &= -\frac{v^{5/2} \sinh(v^{1/2}\eta/2)[\cosh^2(v^{1/2}\eta/2) - 3]}{2 \cosh^5(v^{1/2}\eta/2)}; \quad \dots \end{aligned} \quad (42)$$

Thus the boundary conditions for realization of the solitary wave at $\eta = 0$ should be

$$U(0) = \frac{v}{2}; \quad \frac{dU}{d\eta} \big|_{\eta=0} = 0; \quad \frac{d^2U}{d\eta^2} \big|_{\eta=0} = -\frac{v^2}{4}; \quad \frac{d^3U}{d\eta^3} \big|_{\eta=0} = 0; \quad \dots \quad (43)$$

The realization of these boundary conditions is not very probable as the heart pulsations are slightly irregular with respect to amplitude and time between the beats. Thus if a solitary wave solution is realized for a pulsation the next pulsation will lead to slight change of the boundary conditions and the next wave will be not solitary. Then another scenario for blood waves is more probable and this scenario is connected to the periodic solutions of Eq.(35).

The periodic wave solutions of Eq.(35) can be realized for much more values of the boundary conditions and for different amplitudes of the blood waves. Several examples of periodic solutions of Eq.(35) obtained through the system of equation (36) are shown in Fig. 1.

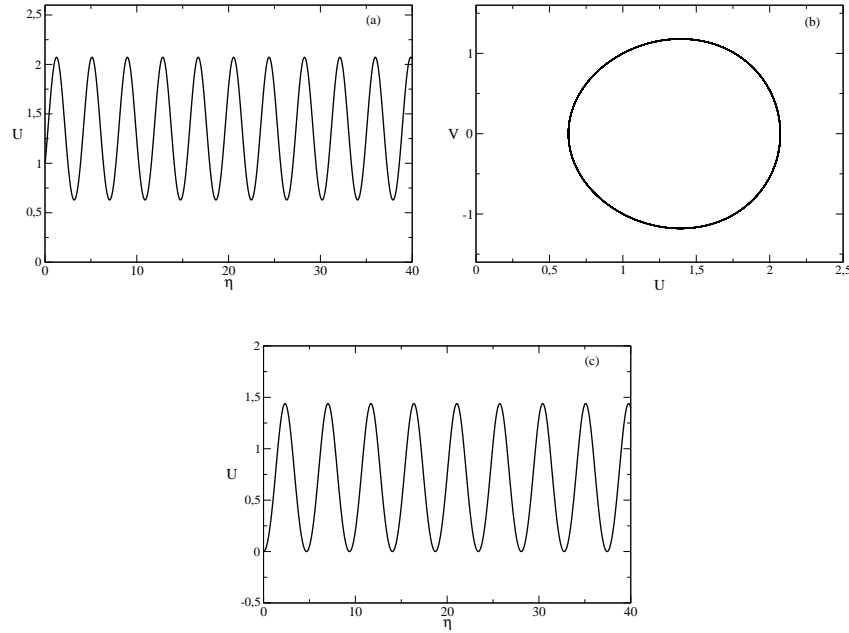


Figure 1: Examples for periodic wave solutions of Eq.(36). Figure (a): $\mu_1 = 1.43$; $\mu_3 = 1.1$. $U(0) = V(0) = W(0) = 0.99$. Figure (b): Diagram of the components U and V for the solution with parameters and initial conditions that are the same as those in Figure (a). Figure(c): $\mu_1 = 4.0$; $\mu_3 = 2.1$. $U(0) = V(0) = 0$; $W(0) = 1$. One may observe that the period and the amplitude of this solution are different with respect to the period and amplitude of the solution from Figure (a).

Because of their larger parameter regions of existence the periodic solutions can be used for construction of the wave motion in the blood in presence of irregularity of heart beats as follows. Let the heart makes a pulsation and the blood wave start to propagate in the artery. This can be modelled by a half a period of the periodic wave solution of Eq.(35). When the next pulsation comes one can stop at the corresponding values of U and its derivatives and can treat them as the new initial conditions. These new initial conditions describe slightly different periodic wave. Half a period of this wave can be used to model the blood flow wave up to the moment of the next pulsation. At this moment the reached values of U and its derivatives are again the initial conditions that describe the blood wave corresponding to the third pulsation, etc. In such a way a sequence of slightly different waves (shown in Fig.2.) may model the slight irregularity of the heart activity. It is well

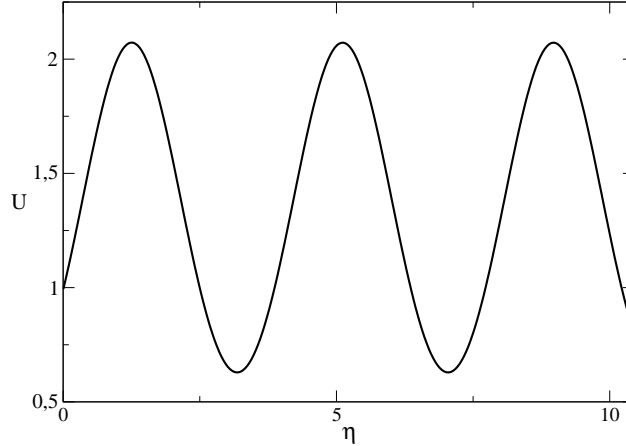


Figure 2: Periodic blood waves constructed by parts of periodic solutions of Eq.(35) by the algorithm described in the text. It may seem that a single wave is shown at the figure. Actually the wave is constructed by the parts of 3 different periodic solutions of Eq.(35). Namely it is assumed that the first pulsation started at $\eta = 0$ with initial conditions $U(0) = V(0) = W(0) = 0.99$ and $\mu_1 = 1.43$; $\mu_3 = 1.1$. The second pulsation was at $\eta = 2.844$ with initial conditions $U(2.844) = 0.7253$; $V(2.844) = -0.5571$; $W(2.844) = 1.5255$ and $\mu_1 = 1.42$; $\mu_3 = 1.11$. The third pulsation was at $\eta = 6.924$ with initial conditions $U(6.924) = 0.6409$; $V(6.924) = -0.2000$; $W(6.924) = 1.6772$ and $\mu_1 = 1.44$; $\mu_3 = 1.08$.

known that the time intervals between the heart pulsations are long-range correlated and this is one of the numerous arising of long-range correlations in various systems [40] - [43]. Such long-range correlated pseudorandom sequences modeling heart activity may be generated by a computer program and the values in the sequence will determine the end of the corresponding wave train and the beginning of the next wave train. This was realized in Fig.2.

5 Concluding remarks

In this paper we have shown that the area of research on blood flow is a large area for application of the methods of nonlinear dynamics. Even the relative single problems such as investigation of waves in large arteries (where the fluid can be treated as Newtonian and the long-wave approximation significantly simplifies the equations) lead to relatively complicated model equations such as the discussed above variable coefficient KdV equation. We have stressed that the solitary wave solution of the model equation requires very specific initial conditions and relationship among the two model parameters. Because of this the probability for realization of this solution is small. We have discussed another kind of solutions of the nonlinear model equation that are much more probable for realization: they do not require relationship between the two parameters of the model and are robust against change of the initial conditions due to the irregularities of the pulsation dynamics of the heart. These solutions are the periodic solutions of the system of equations (36). Using parts of these solutions one can construct a model profile of a blood waves that reflect the irregularities and the long-range correlations presented in the pulsation activity of human heart.

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